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SOLUTIONS OF THE EQUATIONS OF EQUILIBRIUM OF ELASTIC DIELECTRICS: STRESS FUNCTIONS, CONCENTRATED FORCE, SURFACE ENERGY*

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Abstract—In this paper, functions analogous to the Papkovitch functions of classical elasticity are derived for Mindlin's linear theory of elastic dielectrics whose energy density of deformation and polarization depends on the gradient of the polarization, as well as on the strain and on the polarization itself. These functions are then used to solve the problem of the concentrated force. They are also used to solve the problems of the sphere and of the spherical cavity, in the absence of all external actions, in order to display the influence of surface curvature on the surface energy of deformation and polarization inherent in the theory.

1. INTRODUCTION

IN A recent paper, Mindlin [1] extended a linear version of Toupin's [2] form of the classical equations of an elastic dielectric to include the contribution of the polarization gradient to the stored energy of deformation and polarization. The main effects of the augmentation of the equations are the accommodation of a surface energy of deformation and polarization and the introduction of a new, linear, electro-mechanical effect in both non-centrosymmetric and centrosymmetric (including isotropic) materials.

The present paper is concerned with Mindlin's equations of equilibrium for isotropic dielectrics. A general solution is found in terms of functions analogous to the Papkovitch functions of classical elasticity and particular solutions are given for the concentrated force and for the surface energies of deformation and polarization at internal and external spherical surfaces.

As in Kelvin's solution in classical elasticity, the singularity of the displacement field for the concentrated force is of order r^{-1}, as are the singularities of the polarization and Maxwell electric self-field.

It is also found that the surface energy of deformation and polarization for a plane surface cannot be a minimum compared to those of curved surfaces; that is, the surface energies of deformation and polarization of an internal and an external spherical surface cannot both be greater than that of a plane surface of the same material, although either one can be either less or greater. Whatever the case, the effect of surface curvature is quite small in the range of curvatures in which the equations are expected to be valid.

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2. THE EQUATIONS OF AN ELASTIC DIELECTRIC WITH POLARIZATION GRADIENT

Let the body occupy a region V, whose boundary S separates it from an outer vacuum V'. If the body is in static equilibrium, the field equations developed in [1] reduce to

$$T_{ij,i}+f_j = 0, \tag{2.1a}$$

$$\bar{E}_j+E_{ij,i}-\varphi_{,j}+E_j^0 = 0, \tag{2.1b}$$

$$-\varepsilon_0\varphi_{,ii}+P_{i,i} = 0, \quad \text{in } V, \tag{2.1c}$$

$$\varphi_{,ii} = 0, \quad \text{in } V'; \tag{2.1d}$$

and, on a free boundary S,

$$n_iT_{ij} = 0, \tag{2.2a}$$

$$n_iE_{ij} = 0, \tag{2.2b}$$

$$n_i(-\varepsilon_0[\varphi_{,i}]+P_i) = 0. \tag{2.2c}$$

In the equations quoted so far, f_j and E_j^0 are the external body force and electric field, φ is the potential of the Maxwell self-field; i.e.

$$E_i^{MS} = -\varphi_{,i}, \tag{2.3}$$

P_i is the polarization, $[\varphi_{,i}]$ is the jump in $\varphi_{,i}$ across S, n_i is the unit normal to S, directed outward from V, and ε_0 is the permittivity of a vacuum.

Constitutive relations for the stress T_{ij}, the effective local electric force $\bar{E}_j$, and E_{ij} are derived from the energy density W^L of deformation and polarization. For an isotropic and centrosymmetric material, W^L is given by

$$\begin{aligned} W^L = {} & b_0P_{i,i}+\tfrac{1}{2}aP_iP_i+\tfrac{1}{2}b_{12}P_{i,i}P_{j,j}+\tfrac{1}{2}(b_{44}+b_{77})P_{j,i}P_{j,i} \\ & +\tfrac{1}{2}(b_{44}-b_{77})P_{j,i}P_{i,j}+\tfrac{1}{2}c_{12}S_{ii}S_{jj}+c_{44}S_{ij}S_{ij} \\ & +d_{12}P_{i,i}S_{jj}+2d_{44}P_{j,i}S_{ij}, \end{aligned} \tag{2.4}$$

where S_{ij} is the strain, given by

$$S_{ij} = \tfrac{1}{2}(u_{j,i}+u_{i,j}), \tag{2.5}$$

in which u_j is the displacement. Then

$$-\bar{E}_j \equiv \frac{\partial W^L}{\partial P_j} = aP_j, \tag{2.6a}$$

$$\begin{aligned} E_{ij} \equiv \frac{\partial W^L}{\partial P_{j,i}} = {} & b_{12}\delta_{ij}P_{k,k}+(b_{44}+b_{77})P_{j,i} \\ & +(b_{44}-b_{77})P_{i,j}+d_{12}\delta_{ij}S_{kk}+2d_{44}S_{ij}+b_0\delta_{ij}, \end{aligned} \tag{2.6b}$$

$$\begin{aligned} T_{ij} \equiv \frac{\partial W^L}{\partial S_{ij}} = {} & d_{12}\delta_{ij}P_{k,k}+d_{44}(P_{j,i}+P_{i,j}) \\ & +c_{12}\delta_{ij}S_{kk}+2c_{44}S_{ij} = T_{ji}, \end{aligned} \tag{2.6c}$$

where S_{ij} and S_{ji}, $i \neq j$, are considered to be independent variables when $\partial W^L/\partial S_{ij}$ are formed.

For a homogeneous material, substitution of (2.5) into (2.6), and subsequent substitution of the result into (2.1), give the "displacement" equations of equilibrium. In vector form, these are [1]

$$c_{44}\nabla^2\mathbf{u}+(c_{12}+c_{44})\nabla\nabla\cdot\mathbf{u}+d_{44}\nabla^2\mathbf{P}+(d_{12}+d_{44})\nabla\nabla\cdot\mathbf{P}+\mathbf{f}=0, \tag{2.7a}$$

$$d_{44}\nabla^2\mathbf{u}+(d_{12}+d_{44})\nabla\nabla\cdot\mathbf{u}+(b_{44}+b_{77})\nabla^2\mathbf{P}$$
$$+(b_{12}+b_{44}-b_{77})\nabla\nabla\cdot\mathbf{P}-a\mathbf{P}-\nabla\varphi+\mathbf{E}^0=0, \tag{2.7b}$$

$$-\varepsilon_0\nabla^2\varphi+\nabla\cdot\mathbf{P}=0, \quad \text{in } V, \tag{2.7c}$$

$$\nabla^2\varphi=0, \quad \text{in } V'. \tag{2.7d}$$

Substitution of (2.5) into (2.6), and subsequent substitution of the result into (2.2), give the boundary conditions for a free surface S in terms of $\mathbf{u}$, $\mathbf{P}$, and φ. In vector form these are

$$d_{12}\mathbf{n}\nabla\cdot\mathbf{P}+d_{44}\mathbf{n}\cdot(\nabla\mathbf{P}+\mathbf{P}\nabla)+c_{12}\mathbf{n}\nabla\cdot\mathbf{u}+c_{44}\mathbf{n}\cdot(\nabla\mathbf{u}+\mathbf{u}\nabla)=0, \tag{2.8a}$$

$$b_{12}\mathbf{n}\nabla\cdot\mathbf{P}+b_{44}\mathbf{n}\cdot(\nabla\mathbf{P}+\mathbf{P}\nabla)+b_{77}\mathbf{n}\cdot(\nabla\mathbf{P}-\mathbf{P}\nabla)$$
$$+d_{12}\mathbf{n}\nabla\cdot\mathbf{u}+d_{44}\mathbf{n}\cdot(\nabla\mathbf{u}+\mathbf{u}\nabla)+b_0\mathbf{n}=0, \tag{2.8b}$$

$$\mathbf{n}\cdot(-\varepsilon_0[\nabla\varphi]+\mathbf{P})=0. \tag{2.8c}$$

The density W of the total self energy is given by

$$W=\begin{cases} W^L+\frac{1}{2}\varepsilon_0\varphi_{,i}\varphi_{,i}, & \text{in } V, \\ \frac{1}{2}\varepsilon_0\varphi_{,i}\varphi_{,i}, & \text{in } V'. \end{cases} \tag{2.9}$$

For a centrosymmetric, isotropic and homogeneous material, the expression, derived in [1], for the total self energy associated with a body satisfying the equations of equilibrium (2.1) without external body force or external electric field, and satisfying the boundary conditions (2.2) for a free boundary, reduces to

$$\int_{V+V'} W\,\mathrm{d}V=\frac{1}{2}b_0\int_S n_iP_i\,\mathrm{d}S<0, \tag{2.10}$$

where the inequality is a consequence of the positive-definiteness of the quadratic part of the energy density W. From (2.10), the surface energy of deformation and polarization per unit area, henceforth called the surface energy, is

$$T=\tfrac{1}{2}b_0[n_iP_i]_S. \tag{2.11}$$

This energy is to be added to the bond energy, per unit area, to obtain the total energy per unit area required to separate the material into two parts along a surface S [3].

3. STRESS FUNCTIONS

In this section it is proved that any solution $\{\mathbf{u}, \mathbf{P}, \varphi\}$ of the "displacement" equations of equilibrium (2.7a, b, c), in a region V bounded by a surface S, can be expressed as

$$\mathbf{u} = \mathbf{B} - \tfrac{1}{2}(1-k)\nabla(\mathbf{r}\cdot\mathbf{B}+B_0) + a^{-1}c_{44}k_2(k_2-k_1)\nabla\nabla\cdot\mathbf{B} - \varepsilon_0 k_1\nabla\varphi + a^{-1}k_2(1+a\varepsilon_0)(1-l_1^2\nabla^2)\nabla\varphi - k_2(\mathbf{K}-l_2^2\nabla\nabla\cdot\mathbf{K}), \tag{3.1a}$$

$$\mathbf{P} = -a^{-1}c_{44}(k_2-k_1)\nabla\nabla\cdot\mathbf{B} + \varepsilon_0\nabla\varphi - a^{-1}(1+a\varepsilon_0)(1-l_1^2\nabla^2)\nabla\varphi + \mathbf{K} - l_2^2\nabla\nabla\cdot\mathbf{K}, \tag{3.1b}$$

provided that $\mathbf{B}$, B_0, $\mathbf{K}$, and φ satisfy, in V, the equations

$$c_{44}\nabla^2\mathbf{B} = -\mathbf{f}, \tag{3.2a}$$

$$c_{44}\nabla^2 B_0 = \mathbf{r}\cdot\mathbf{f}, \tag{3.2b}$$

$$a(1-l_2^2\nabla^2)\mathbf{K} = \mathbf{E}^0 - k_2\mathbf{f}, \tag{3.2c}$$

$$(1+a\varepsilon_0)(1-l_1^2\nabla^2)\nabla^2\varphi = \nabla\cdot\mathbf{E}^0 - k_1\nabla\cdot\mathbf{f}, \tag{3.2d}$$

where $\mathbf{r}$ is the position vector, and

$$k \equiv c_{44}/(c_{12}+2c_{44}), \tag{3.3a}$$

$$k_1 \equiv (d_{12}+2d_{44})/(c_{12}+2c_{44}), \tag{3.3b}$$

$$k_2 \equiv d_{44}/c_{44}, \tag{3.3c}$$

and where

$$l_1^2 \equiv \varepsilon_0[(b_{12}+2b_{44})(c_{12}+2c_{44})-(d_{12}+2d_{44})^2]/(1+a\varepsilon_0)(c_{12}+2c_{44}) > 0, \tag{3.4a}$$

$$l_2^2 \equiv [(b_{44}+b_{77})c_{44}-d_{44}^2]/ac_{44} > 0. \tag{3.4b}$$

The inequalities of (3.4) are necessary conditions for the positive-definiteness of the quadratic part of the energy density W. Each of the parameters l_1 and l_2 has the dimension of length.

A proof of completeness of the representation (3.1) will now be given. First, by means of the usual proof of the Helmholtz resolution, functions φ_1 and $\mathbf{H}_1$ are constructed from $\mathbf{u}$, by means of Poisson integrals, so that $\mathbf{u}$ has the representation

$$\mathbf{u} = \varepsilon_0\nabla\varphi_1 + \nabla\times\mathbf{H}_1, \qquad \nabla\cdot\mathbf{H}_1 = 0. \tag{3.5}$$

Next, it is noted that (2.7c) may be written as

$$\nabla\cdot(-\varepsilon_0\nabla\varphi+\mathbf{P}) = 0.$$

Now, for any solenoidal vector $\mathbf{v}$ there exists a vector $\mathbf{w}$, such that

$$\mathbf{v} = \nabla\times\mathbf{w}.$$

Thus, in the present case, there exists a vector $\mathbf{H}_2$ such that

$$-\varepsilon_0\nabla\varphi + \mathbf{P} = \nabla\times\mathbf{H}_2,$$

so that $\mathbf{P}$ has the representation

$$\mathbf{P} = \varepsilon_0\nabla\varphi + \nabla\times\mathbf{H}_2. \tag{3.6}$$

The equations (3.5) and (3.6) are substituted into (2.7a, b, c). Equation (2.7c) is satisfied automatically, while (2.7a) and (2.7b) assume, respectively, the forms

$$\varepsilon_0\nabla^2\nabla[(c_{12}+2c_{44})\varphi_1+(d_{12}+2d_{44})\varphi]+\nabla^2\nabla\times[c_{44}\mathbf{H}_1+d_{44}\mathbf{H}_2]+\mathbf{f}=0, \tag{3.7a}$$

$$\begin{aligned}\varepsilon_0\nabla\{(d_{12}+2d_{44})\nabla^2\varphi_1+[(b_{12}+2b_{44})\nabla^2-(a+\varepsilon_0^{-1})]\varphi\}\\ +\nabla\times\{d_{44}\nabla^2\mathbf{H}_1+[(b_{44}+b_{77})\nabla^2-a]\mathbf{H}_2\}+\mathbf{E}^0=0.\end{aligned} \tag{3.7b}$$

In order to assure completeness of the representation for every set of material constants compatible with positive-definiteness of the quadratic part of the energy density W, division by $d_{12}+2d_{44}$ and by d_{44} is prohibited, since each of these constants may be equal to zero within the confines of positive-definiteness. With this in mind, define

$$\psi\equiv\varphi_1+k_1\varphi, \tag{3.8a}$$

$$\mathbf{G}\equiv\mathbf{H}_1+k_2\mathbf{H}_2. \tag{3.8b}$$

Substitution of (3.8) into (3.7a) gives

$$c_{44}\nabla^2(k^{-1}\varepsilon_0\nabla\psi+\nabla\times\mathbf{G})+\mathbf{f}=0. \tag{3.9}$$

The functions ψ and $\mathbf{G}$ are now resolved into Papkovitch functions by the procedure of [4].

Define

$$\mathbf{B}\equiv k^{-1}\varepsilon_0\nabla\psi+\nabla\times\mathbf{G}. \tag{3.10}$$

Substitution of (3.10) into (3.9) yields the desired differential equation (3.2a). Also, the divergence of (3.10) is

$$\nabla\cdot\mathbf{B}=k^{-1}\varepsilon_0\nabla^2\psi. \tag{3.11}$$

Define

$$B_0\equiv 2k^{-1}\varepsilon_0\psi-\mathbf{r}\cdot\mathbf{B}. \tag{3.12}$$

With the aid of (3.11) and the already established (3.2a), it is found that B_0 satisfies the desired differential equation (3.2b).

Equation (3.12) may be used to eliminate ψ from (3.8a). Thus,

$$\varphi_1=\psi-k_1\varphi=\tfrac{1}{2}k\varepsilon_0^{-1}(\mathbf{r}\cdot\mathbf{B}+B_0)-k_1\varphi. \tag{3.13}$$

Equations (3.12) and (3.10) may be used to eliminate $\mathbf{G}$ from (3.8b). Thus,

$$\nabla\times\mathbf{H}_1=\nabla\times\mathbf{G}-k_2\nabla\times\mathbf{H}_2=\mathbf{B}-\tfrac{1}{2}\nabla(\mathbf{r}\cdot\mathbf{B}+B_0)-k_2\nabla\times\mathbf{H}_2. \tag{3.14}$$

Substitution of (3.14) and (3.13) into (3.7b) gives

$$(1+a\varepsilon_0)(1-l_1^2\nabla^2)\nabla\varphi+a(1-l_2^2\nabla^2)\nabla\times\mathbf{H}_2+c_{44}(k_2-k_1)\nabla\nabla\cdot\mathbf{B}-\mathbf{E}^0+k_2\mathbf{f}=0, \tag{3.15}$$

with the aid of the already established (3.2a) and (3.2b).

Now, following a procedure similar to that of [4], define

$$\begin{aligned}4\pi l_2^2\mathbf{K}'\equiv a^{-1}\int_V r_1^{-1}\mathrm{e}^{-r_1/l_2}[(1+a\varepsilon_0)(1-l_1^2\nabla^2)\nabla\varphi\\ +a(1-l_2^2\nabla^2)\nabla\times\mathbf{H}_2+c_{44}(k_2-k_1)\nabla\nabla\cdot\mathbf{B}]_Q\,\mathrm{d}V_Q,\end{aligned} \tag{3.16}$$

where r_1 is the distance between a field point $P(x, y, z)$ and a source point $Q(\xi, \eta, \zeta)$; and $dV_Q = d\eta\, d\xi\, d\zeta$. Then by a process similar to that for Poisson's equation [5],

$$a(1-l_2^2\nabla^2)\mathbf{K}' = (1+a\varepsilon_0)(1-l_1^2\nabla^2)\nabla\varphi + a(1-l_2^2\nabla^2)\nabla\times\mathbf{H}_2 + c_{44}(k_2-k_1)\nabla\nabla\cdot\mathbf{B}. \quad (3.17)$$

Comparison with (3.15) shows immediately that

$$a(1-l_2^2\nabla^2)\mathbf{K}' = \mathbf{E}^0 - k_2\mathbf{f}. \quad (3.18)$$

Also, the divergence of (3.17) is

$$a(1-l_2^2\nabla^2)\nabla\cdot\mathbf{K}' = (1+a\varepsilon_0)(1-l_1^2\nabla^2)\nabla^2\varphi + c_{44}(k_2-k_1)\nabla^2\nabla\cdot\mathbf{B}. \quad (3.19)$$

Now define

$$\mathbf{K}'' \equiv \nabla\times\mathbf{H}_2 - \mathbf{K}' + l_2^2\nabla\nabla\cdot\mathbf{K}' + a^{-1}(1+a\varepsilon_0)(1-l_1^2\nabla^2)\nabla^2\varphi + a^{-1}c_{44}(k_2-k_1)\nabla\nabla\cdot\mathbf{B}. \quad (3.20)$$

With the aid of (3.17) and (3.19), it is seen that $\mathbf{K}''$ satisfies

$$\nabla\cdot\mathbf{K}'' = 0, \qquad (1-l_2^2\nabla^2)\mathbf{K}'' = 0. \quad (3.21)$$

The definition (3.20) provides a representation for $\nabla\times\mathbf{H}_2$; viz.,

$$\nabla\times\mathbf{H}_2 = \mathbf{K}' + \mathbf{K}'' - l_2^2\nabla\nabla\cdot\mathbf{K}' - a^{-1}(1+a\varepsilon_0)(1-l_1^2\nabla^2)\nabla\varphi + a^{-1}c_{44}(k_1-k_2)\nabla\nabla\cdot\mathbf{B}. \quad (3.22)$$

Now define

$$\mathbf{K} \equiv \mathbf{K}' + \mathbf{K}''. \quad (3.23)$$

By virtue of (3.21), the representation (3.22) for $\nabla\times\mathbf{H}_2$ becomes

$$\nabla\times\mathbf{H}_2 = \mathbf{K} - l_2^2\nabla\nabla\cdot\mathbf{K} - a^{-1}(1+a\varepsilon_0)(1-l_1^2\nabla^2)\nabla\varphi + a^{-1}c_{44}(k_1-k_2)\nabla\nabla\cdot\mathbf{B}. \quad (3.24)$$

By virtue of (3.21) and (3.18), $\mathbf{K}$ satisfies the desired differential equation (3.2c), while (3.2d) results from taking the divergence of (3.15), and using the previously established (3.2a) to eliminate $\nabla^2\mathbf{B}$.

The desired representation (3.1b) for $\mathbf{P}$ is found by substituting (3.24) into (3.6). In order to find the desired representation (3.1a) for $\mathbf{u}$, (3.24) is first substituted into (3.14) to obtain

$$\begin{aligned}\nabla\times\mathbf{H}_1 = {} & \mathbf{B} - \tfrac{1}{2}\nabla(\mathbf{r}\cdot\mathbf{B} + B_0) \\ & - k_2[\mathbf{K} - l_2^2\nabla\nabla\cdot\mathbf{K} - a^{-1}(1+a\varepsilon_0)(1-l_1^2\nabla^2)\nabla\varphi + a^{-1}c_{44}(k_1-k_2)\nabla\nabla\cdot\mathbf{B}].\end{aligned} \quad (3.25)$$

Then (3.25) and (3.13) are substituted into (3.5) and terms are rearranged, thereby obtaining the representation (3.1a) for $\mathbf{u}$.

It has thus been shown that if $\{\mathbf{u}, \mathbf{P}, \varphi\}$ is a solution of (2.7a, b, c), then $\mathbf{u}$ and $\mathbf{P}$ have the representation (3.1), and $\mathbf{B}$, B_0, $\mathbf{K}$, φ satisfy the equations (3.2).

The converse, that $\{\mathbf{u}, \mathbf{P}, \varphi\}$ is a solution of (2.7a, b, c) if the equations of (3.2) are satisfied, is shown by straightforward substitution of (3.1) into the left hand sides of (2.7a, b, c), under the conditions (3.2).

4. CONCENTRATED FORCE IN A REGION OF INFINITE EXTENT

In a region V of infinite extent, let the body force be zero outside a finite region V_0, which contains the origin and a non-vanishing field of unidirectional forces $\mathbf{f}$, and let the external electric field $\mathbf{E}^0$ be zero everywhere. A concentrated force is defined by

$$\mathbf{F} \equiv \lim_{V_0 \to 0} \int_V \mathbf{f}_Q \, \mathrm{d}V_Q. \tag{4.1}$$

In [6] it was shown that in an infinite region, solutions of equations of the types found in (3.2) are

$$4\pi c_{44}\mathbf{B} = \int_V r_1^{-1}\mathbf{f}_Q \, \mathrm{d}V_Q, \tag{4.2}$$

$$4\pi c_{44}B_0 = -\int_V r_1^{-1}\mathbf{r}' \cdot \mathbf{f}_Q \, \mathrm{d}V_Q, \tag{4.3}$$

$$4\pi a l_2^2 \mathbf{K} = -k_2 \int_V r_1^{-1} \, \mathrm{e}^{-r_1/l_2}\mathbf{f}_Q \, \mathrm{d}V_Q, \tag{4.4}$$

$$4\pi(1+a\varepsilon_0)\varphi = k_1 \int_V r_1^{-1}(1-\mathrm{e}^{-r_1/l_1})\nabla_Q \cdot \mathbf{f}_Q \, \mathrm{d}V_Q, \tag{4.5}$$

where $r' = \sqrt{\xi^2+\eta^2+\zeta^2}$. Now

$$\lim_{V_0 \to 0} r_1 = r \quad \text{and} \quad \lim_{V_0 \to 0} r' = 0.$$

Hence, for the concentrated force, (4.2) reduces to

$$4\pi c_{44}\mathbf{B} = \frac{\mathbf{F}}{r}, \tag{4.6}$$

(4.3) reduces to

$$B_0 = 0, \tag{4.7}$$

and (4.4) reduces to

$$4\pi a l_2^2 \mathbf{K} = -k_2 r^{-1} \, \mathrm{e}^{-r/l_2}\mathbf{F}. \tag{4.8}$$

If $\mathbf{f}$ is taken to be continuous across the boundary of V_0, then the integrand of (4.5) is transformed as follows:

$$\begin{aligned} 4\pi(1+a\varepsilon_0)\varphi = k_1 \int_S r_1^{-1}(1-\mathrm{e}^{-r_1/l_1})\mathbf{n}_Q \, . \, \mathbf{f}_Q \, \mathrm{d}S_Q \\ -k_1 \int_V \nabla_Q[r_1^{-1}(1-\mathrm{e}^{-r_1/l_1})] \cdot \mathbf{f}_Q \, \mathrm{d}V_Q. \end{aligned} \tag{4.9}$$

The surface integral in (4.9) vanishes because $\mathbf{f} = 0$ outside V_0. Also,

$$\lim_{V_0 \to 0} \int_V \nabla_Q[r_1^{-1}(1-\mathrm{e}^{-r_1/l_1})] \cdot \mathbf{f}_Q \, \mathrm{d}V_Q = -\mathbf{F} \cdot \nabla[r^{-1}(1-\mathrm{e}^{-r/l_1})]. \tag{4.10}$$

Hence,

$$4\pi(1+a\varepsilon_0)\varphi = k_1\mathbf{F}\cdot\nabla[r^{-1}(1-e^{-r/l_1})]. \tag{4.11}$$

Equations (4.6), (4.7), (4.8), and (4.11) constitute the solution of (2.7a, b, c) for the concentrated force.

Substitution of (4.11) into (2.3) yields

$$4\pi\mathbf{E}^{MS} = \frac{k_1}{(1+a\varepsilon_0)l_1^2}\left[\frac{\mathbf{F}}{r}\psi\left(\frac{r}{l_1}\right) - \frac{3\mathbf{rr}\cdot\mathbf{F}}{r^3}\psi\left(\frac{r}{l_1}\right) + \frac{\mathbf{rr}\cdot\mathbf{F}}{r^3}e^{-r/l_1}\right] \tag{4.12a}$$

as the Maxwell self-field, while substitution of (4.6), (4.7), (4.8), and (4.11) into (3.1) gives the displacement and polarization as

$$\begin{aligned} 4\pi\mathbf{u} = {} & \frac{1+k}{2c_{44}}\frac{\mathbf{F}}{r} + \frac{1-k}{2c_{44}}\frac{\mathbf{rr}\cdot\mathbf{F}}{r^3} \\ & + \frac{\varepsilon_0 k_1^2}{(1+a\varepsilon_0)l_1^2}\left[\frac{\mathbf{F}}{r}\psi\left(\frac{r}{l_1}\right) - \frac{3\mathbf{rr}\cdot\mathbf{F}}{r^3}\psi\left(\frac{r}{l_1}\right) + \frac{\mathbf{rr}\cdot\mathbf{F}}{r^3}e^{-r/l_1}\right] \\ & + \frac{\varepsilon_0 k_2^2}{a\varepsilon_0 l_2^2}\left[-\frac{\mathbf{F}}{r}\psi\left(\frac{r}{l_2}\right) + \frac{\mathbf{F}}{r}e^{-r/l_2} + \frac{3\mathbf{rr}\cdot\mathbf{F}}{r^3}\psi\left(\frac{r}{l_2}\right) - \frac{\mathbf{rr}\cdot\mathbf{F}}{r^3}e^{-r/l_2}\right], \end{aligned} \tag{4.12b}$$

$$\begin{aligned} 4\pi\mathbf{P} = {} & \frac{-\varepsilon_0 k_1}{(1+a\varepsilon_0)l_1^2}\left[\frac{\mathbf{F}}{r}\psi\left(\frac{r}{l_1}\right) - \frac{3\mathbf{rr}\cdot\mathbf{F}}{r^3}\psi\left(\frac{r}{l_1}\right) + \frac{\mathbf{rr}\cdot\mathbf{F}}{r^3}e^{-r/l_1}\right] \\ & - \frac{\varepsilon_0 k_2}{a\varepsilon_0 l_2^2}\left[-\frac{\mathbf{F}}{r}\psi\left(\frac{r}{l_2}\right) + \frac{\mathbf{F}}{r}e^{-r/l_2} + \frac{3\mathbf{rr}\cdot\mathbf{F}}{r^3}\psi\left(\frac{r}{l_2}\right) - \frac{\mathbf{rr}\cdot\mathbf{F}}{r^3}e^{-r/l_2}\right], \end{aligned} \tag{4.12c}$$

where

$$\psi(x) \equiv x^{-2} - x^{-1}e^{-x} - x^{-2}e^{-x}. \tag{4.13}$$

Now,

$$\lim_{x\to 0}\psi(x) = \tfrac{1}{2}.$$

Therefore, $|\mathbf{E}^{MS}|$, $|\mathbf{u}|$, and $|\mathbf{P}|$ all behave as r^{-1} as $r \to 0$.

5. SOLUTION FOR THE SPHERICAL CAVITY

Let r be the length of the position vector $\mathbf{r}$, and r_c the radius of a spherical cavity in a body of infinite extent. Then V is the region $r > r_c$, V' is the region $r < r_c$, and S is the closed surface defined by $r = r_c$. The solution of (2.7) and (2.8), bounded everywhere, vanishing as $r \to \infty$, and for which φ is continuous at $r = r_c$, is to be found for the case where $\mathbf{f} = 0$ and $\mathbf{E}^0 = 0$.

Suitable solutions of (2.7) are given by the stress functions

$$B_0 = A_1 r^{-1}, \tag{5.1a}$$

$$\varphi = A_2 r^{-1}\, e^{-r/l_1}, \tag{5.1b}$$

$$\mathbf{B} = \mathbf{K} = 0, \qquad r > r_c, \tag{5.1c}$$

$$\varphi = A_2 r_c^{-1}\, e^{-r_c/l_1}, \qquad r < r_c, \tag{5.1d}$$

where A_1 and A_2 are constants.

Equation (3.1) then gives the displacement and polarization as

$$\mathbf{u} = -\tfrac{1}{2}(1-k)A_1\nabla\left(\frac{1}{r}\right) - k_1\varepsilon_0 A_2 \nabla(r^{-1}\, e^{-r/l_1}), \tag{5.2a}$$

$$\mathbf{P} = \varepsilon_0 A_2 \nabla(r^{-1}\, e^{-r/l_1}), \qquad r > r_c. \tag{5.2b}$$

With the insertion of (5.1b), (5.1d), and (5.2), the boundary condition (2.8c) at $r = r_c$ is satisfied identically, while (2.8a) and (2.8b) reduce to

$$(1-k)A_1 + 2(k_1 - k_2)\varepsilon_0(\rho_c + 1)\, e^{-\rho_c} A_2 = 0 \tag{5.3a}$$

$$-2(1-k)k_2 c_{44} A_1 + \{(1+a\varepsilon_0)l_1^2\rho_c^2 + 4\varepsilon_0[a l_0^2 - (k_1-k_2)k_2 c_{44}](\rho_c+1)\}e^{-\rho_c}A_2 = -b_0 l_1^3 \rho_c^3, \tag{5.3b}$$

where

$$\rho_c \equiv r_c/l_1, \tag{5.4a}$$

$$l_0^2 \equiv (b_{44}c_{44} - d_{44}^2)/a c_{44} > 0, \tag{5.4b}$$

in which the inequality is a consequence of positive-definiteness of the quadratic part of W. The solution of (5.3) for A_1 and A_2 is

$$A_1 = \frac{-2b_0\varepsilon_0(k_1-k_2)l_1\rho_c^3(\rho_c+1)}{(1-k)(1+a\varepsilon_0)[\rho_c^2 + \alpha^2(\rho_c+1)]}, \tag{5.5a}$$

$$A_2 = \frac{-b_0 l_1 \rho_c^3\, e^{\rho_c}}{(1+a\varepsilon_0)[\rho_c^2 + \alpha^2(\rho_c+1)]}, \tag{5.5b}$$

where

$$\alpha^2 \equiv 4a\varepsilon_0 l_0^2/(1+a\varepsilon_0)l_1^2. \tag{5.6}$$

The stress functions given by (5.1), with the constants A_1 and A_2 given by (5.5), are the solution for the spherical cavity.

6. SOLUTION FOR THE SPHERE

Let r_s be the radius of a solid sphere. Then V is the region $r < r_s$, V' is the region $r > r_s$, and S is the closed surface defined by $r = r_s$. The solution of (2.7) and (2.8), bounded everywhere, vanishing as $r \to \infty$, and in which φ is continuous at $r = r_s$, is to be found for the case where $\mathbf{f} = 0$ and $\mathbf{E}^0 = 0$.

Suitable solutions of (2.7) are given by the stress functions

$$\mathbf{B} = A'_1\mathbf{r}, \tag{6.1a}$$

$$\varphi = A'_2 r^{-1}\sinh(r/l_1) - A'_2 r_s^{-1}\sinh(r_s/l_1), \tag{6.1b}$$

$$B_0 = 0, \qquad \mathbf{K} = 0, \qquad r < r_s, \tag{6.1c}$$

$$\varphi = 0, \qquad r > r_s, \tag{6.1d}$$

where A'_1 and A'_2 are constants.

Equation (3.1) then gives the displacement and polarization as

$$\mathbf{u} = kA'_1\mathbf{r} - k_1\varepsilon_0 A'_2\nabla[r^{-1}\sinh(r/l_1)], \tag{6.2a}$$

$$\mathbf{P} = \varepsilon_0 A'_2\nabla[r^{-1}\sinh(r/l_1)], \qquad r < r_s. \tag{6.2b}$$

With the insertion of (6.1b), (6.1d), and (6.2), the boundary condition (2.8c) at $r = r_s$ is satisfied identically, while (2.8a) and (2.8b) reduce to

$$(3-4k)l_1^3\rho_s^3A'_1 + 4\varepsilon_0(k_1-k_2)(\rho_s\cosh\rho_s - \sinh\rho_s)A'_2 = 0, \tag{6.3a}$$

$$(3k_1-4k_2k)c_{44}l_1^3\rho_s^3A'_1 + \{(1+a\varepsilon_0)l_1^2\rho_s^2\sinh\rho_s - 4\varepsilon_0[al_0^2-(k_1-k_2)k_2c_{44}](\rho_s\cosh\rho_s-\sinh\rho_s)\}A'_2 = -b_0l_1^3\rho_s^3, \tag{6.3b}$$

where

$$\rho_s \equiv r_s/l_1. \tag{6.4}$$

The solution of (6.3) for A'_1 and A'_2 is

$$A'_1 = \frac{4\varepsilon_0 b_0(k_1-k_2)(\rho_s\cosh\rho_s-\sinh\rho_s)}{(3-4k)(1+a\varepsilon_0)l_1^2[\rho_s^2\sinh\rho_s - \beta^2(\rho_s\cosh\rho_s-\sinh\rho_s)]}, \tag{6.5a}$$

$$A'_2 = \frac{-b_0l_1\rho_s^3}{(1+a\varepsilon_0)[\rho_s^2\sinh\rho_s - \beta^2(\rho_s\cosh\rho_s-\sinh\rho_s)]}, \tag{6.5b}$$

where

$$\beta^2 \equiv 4\varepsilon_0[(3-4k)al_0^2 + 3(k_1-k_2)^2c_{44}]/(3-4k)(1+a\varepsilon_0)l_1^2. \tag{6.6}$$

The stress functions given by (6.1), with the constants A'_1 and A'_2 given by (6.5), are the solution for the solid sphere.

7. SURFACE ENERGY OF DEFORMATION AND POLARIZATION

The surface energy for a spherical cavity is found by insertion of (5.2b), along with (5.5b), into (2.11). Thus, for the cavity,

$$T(\rho_c) = -\frac{\varepsilon_0 b_0^2\rho_c(\rho_c+1)}{2(1+a\varepsilon_0)l_1[\rho_c^2+\alpha^2(\rho_c+1)]} < 0, \tag{7.1}$$

while insertion of (6.2b), along with (6.5b), into (2.11) gives

$$T(\rho_s) = -\frac{\varepsilon_0 b_0^2\rho_s(\rho_s\cosh\rho_s-\sinh\rho_s)}{2(1+a\varepsilon_0)l_1[\rho_s^2\sinh\rho_s-\beta^2(\rho_s\cosh\rho_s-\sinh\rho_s)]} < 0 \tag{7.2}$$

as the surface energy of the sphere. In addition, the surface energy of a plane surface, found in [1] for a centrosymmetric cubic crystal, becomes

$$T_0 \equiv -\frac{\varepsilon_0 b_0^2}{2(1+a\varepsilon_0)l_1} < 0 \tag{7.3}$$

in the present isotropic case. The inequalities of (7.1), (7.2), and (7.3) are the direct result of the inequality in (2.10).

Now define

$$\lambda \equiv \begin{cases} -\rho_c^{-1} = -l_1/r_c, & \rho_c > 0, \\ 0, & \text{for a plane surface}, \\ \rho_s^{-1} = l_1/r_s, & \rho_s > 0. \end{cases} \tag{7.4}$$

Substitution of λ for ρ_c and ρ_s permits (7.1), (7.2), and (7.3) to be compacted into

$$\frac{T(\lambda)}{T_0} = \left|\frac{T(\lambda)}{T_0}\right| = \begin{cases} \dfrac{1-\lambda}{1-\alpha^2\lambda(1-\lambda)} & \lambda < 0, \\ 1 & \lambda = 0, \\ \dfrac{1-\lambda\tanh(1/\lambda)}{1-\beta^2\lambda[1-\lambda\tanh(1/\lambda)]}, & \lambda > 0. \end{cases} \tag{7.5}$$

The present continuum theory is concerned only with macroscopic spheres and cavities. At the same time, it is supposed here that l_1 is of the order of an intermolecular distance. Therefore, interest lies only in the region

$$\lambda \ll 1. \tag{7.6}$$

Within this region (7.5) exhibits the behavior

$$\left|\frac{T(\lambda)}{T_0}\right| = \begin{cases} 1+(\alpha^2-1)\lambda+O(\lambda^2), & \lambda \to 0^-, \\ 1, & \lambda = 0, \\ 1+(\beta^2-1)\lambda+O(\lambda^2), & \lambda \to 0^+. \end{cases} \tag{7.7}$$

This behavior depends on the restrictions placed on α^2 and β^2 by the requirement that the energy density W be positive-definite. These restrictions are given by

$$0 < \alpha^2 < 3, \tag{7.8a}$$

$$0 < \beta^2 < 3, \tag{7.8b}$$

$$\beta^2-\alpha^2 = 12\varepsilon_0(k_1-k_2)^2 c_{44}/(3-4k)(1+a\varepsilon_0)l_1^2 \geq 0. \tag{7.8c}$$

Within these restrictions there are divers possibilities. Among these are the possibilities that the surface energy may increase or decrease, from its value for the plane, for an internal or an external spherical surface.

The most notable exclusion is that the absolute value of the surface energy cannot decrease for the internal and external spherical surfaces simultaneously. The surface energy is negative, so the surface energy of the plane cannot be a minimum.

As for the magnitude of the effect of spherical curvature on the surface energy, it is quite small, compared to the surface energy of the plane surface, as long as the radius of curvature of the spherical surface is large in comparison with the material constant l_1.

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REFERENCES

[1] R. D. MINDLIN, Polarization gradient in elastic dielectrics, *Int. J. Solids Struct.* **4**, 637–642 (1968).
[2] R. A. TOUPIN, The elastic dielectric, *J. ration. Mech. Analysis* **5**, 849–915 (1956).
[3] M. P. TOSI, Cohesion of atomic solids in the Born model, *Solid St. Phys.* **16**, 92 (1964).
[4] R. D. MINDLIN, Second gradient of strain and surface tension in linear elasticity, *Int. J. Solids Struct.* **1**, 417–438 (1965).
[5] H. JEFFREYS and B. S. JEFFREYS, *Methods of Mathematical Physics*, 3rd ed., pp. 208–212. Cambridge University Press (1962).
[6] R. D. MINDLIN and H. F. TIERSTEN, Effects of couple-stresses in linear elasticity, *Archs ration. Mech. Analysis* **11**, 415–448 (1962).

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Абстракт—В работе определяются функции, аналогичные функциям Папковича классической упругости, для линейной теории Миндлина упругих диэлектриков. Плотность энергии деформации и поляризация этих диэлектриков зависят от градиента поляризации, а также от самой деформации и поляризации. Затем эти функции исопльзуются для решения задачи концентрической силы. Они также используются для решения задачи сфери и сфериеческой полости, при отсуствии всех внешних явлений, с целью указания влияния кривизны поверхности на поверхностную энергию деформации и поляризации, в рамках этой теории.